TELEPORTATION and DENSE CODING PROTOCOLS

We will study two basic protocols that use entanglement as a ressource for communicating information: "teleportation" & "dense coding" protocols.

First we start with a discussion of entanglement itself.

I. Entanglement

We have here in mind a bipartite system with Hilbert space $\mathcal{H}_{total} = \mathcal{H}_A \otimes \mathcal{H}_B$.

In such space there exist a dichotomic classification of states as "<u>product states</u>" and "<u>entangled states</u>".

<u>Definition</u> Product states are states that can be written in the form

$$|\psi\rangle_{AB} = |\varphi\rangle_A \otimes |\chi\rangle_B$$

for some $|\varphi\rangle_A \in \mathcal{H}_A$, $|\chi\rangle_B \in \mathcal{H}_B$

<u>Definition</u> Entangled states are states that cannot be written in the form of a product as above.

<u>Example:</u> Let $\mathcal{H}_A = \mathbb{C}^2$, $\mathcal{H}_B = \mathbb{C}^2$ say two photons with polarization degree of freedom. In general if $|\psi\rangle \in \mathbb{C}^2_A \otimes \mathbb{C}^2_B$

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

in the computational (or Z basis). Recall

$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1.$$

The following are product state :

$$|00\rangle = |0\rangle \otimes |0\rangle$$
$$|01\rangle = |0\rangle \otimes |1\rangle$$
$$|10\rangle = |1\rangle \otimes |0\rangle$$
$$|11\rangle = |1\rangle \otimes |1\rangle$$

but also if $\alpha_{00} = \alpha_{01} = \alpha_{10} = \alpha_{11} = \frac{1}{2}$

we have :

$$\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) = \frac{|0\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$+ \frac{|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

product.

On the other hand it is easy to show (exercise)

that $\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$ is <u>not</u>

<u>a product state</u>, i.e $\nexists$ $\alpha, \beta, \gamma, \delta$ s.t it can

be written $(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$.

Indeed this would require

$$\alpha\gamma = 1, \quad \alpha\delta = +1, \quad \beta\gamma = 1, \quad \beta\delta = -1$$

$$\Rightarrow \quad \alpha^2 \beta^2 \gamma^2 \delta^2 = -1 \quad \Rightarrow \quad \text{contradiction.}$$

So the state is <u>entangled</u>.

Bell states.

The following four states are the so called Bell states

$$|B_{00}\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right)$$

$$|B_{01}\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle + |10\rangle \right)$$

$$|B_{10}\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle - |11\rangle \right)$$

$$|B_{11}\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle - |10\rangle \right).$$

- They are orthonormal and form a basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ (exercise check!).
- Moreover they are <u>entangled</u> (exercise check!)
- See also <u>"rotation invariant property"</u> <u>Page 7</u>

<u>Generalization to 3 parties</u> (say 3 parties or 3 photon pols)

$\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

We see that the classification of states in product & entangled is more complicated. We may say a state is "<u>fully entangled</u>" if it cannot be written as

$$|\varphi\rangle_A \otimes |\varphi'\rangle_B \otimes |\varphi''\rangle_C \qquad (1)$$

$$|\varphi\rangle_{AB} \otimes |\varphi'\rangle_C \qquad (2)$$

$$|\varphi\rangle_{AC} \otimes |\varphi'\rangle_B \qquad (3)$$

$$|\varphi\rangle_{BC} \otimes |\varphi\rangle_A \qquad (4)$$

If it can be factorized as in (1) it is "fully product". If it can be factorized as in (2) or (3) or (4) and not as in (1) then it is "partially entangled".

Examples of fully entangled state are:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right).$$

in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

Rotation invariant property of Bell states.

Let $|\vartheta\rangle = \cos\vartheta\,|0\rangle + \sin\vartheta\,|1\rangle$ a linear polarization state, and $|\vartheta_\perp\rangle = \sin\vartheta|0\rangle - \cos\vartheta|1\rangle$ the orthogonal state.

It can be checked (exercise) that $\forall\vartheta$:

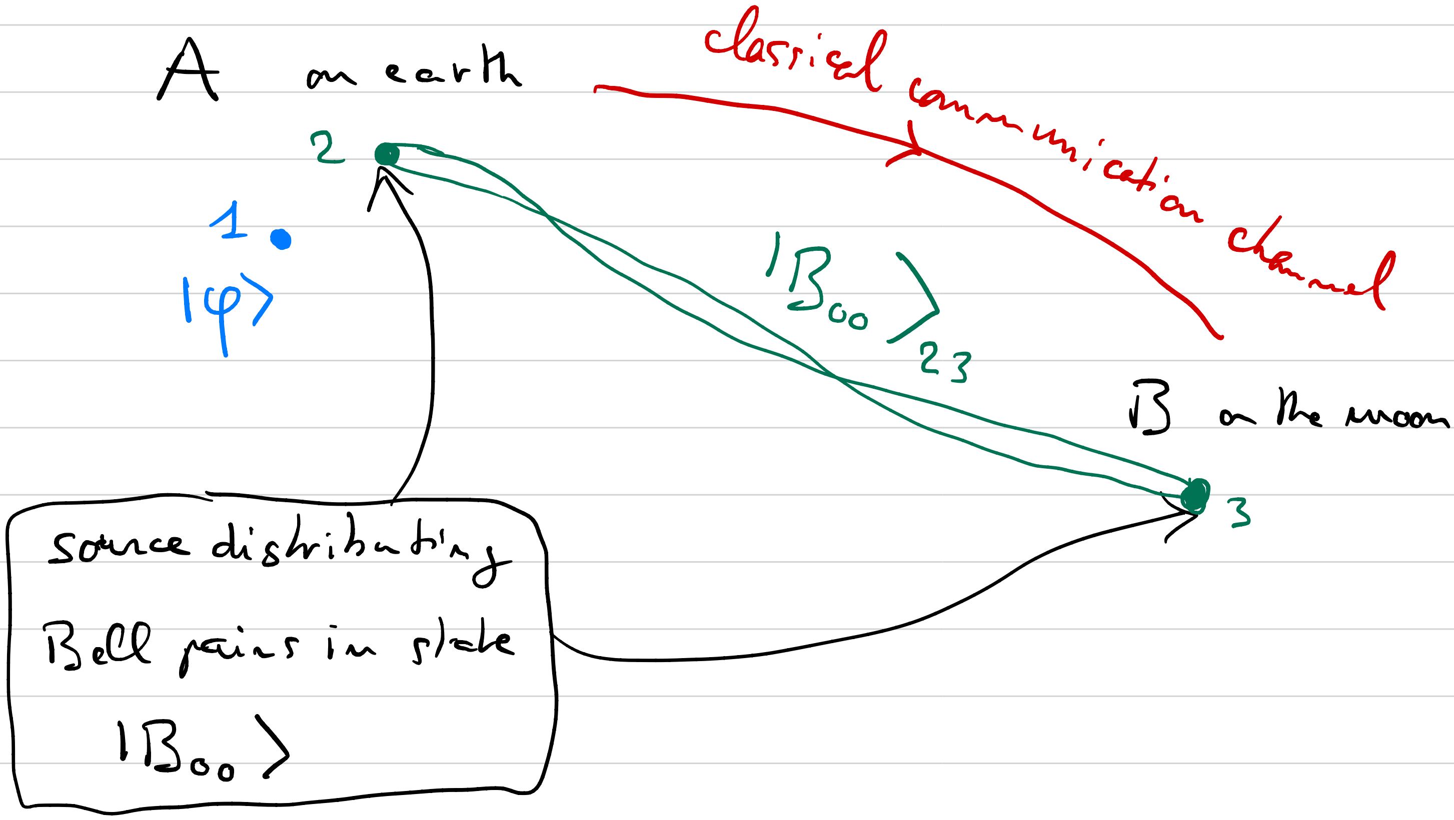
$$\frac{1}{\sqrt{2}}\Big(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle\Big) = \frac{1}{\sqrt{2}}\Big(|\vartheta\rangle\otimes|\vartheta\rangle + |\vartheta_\perp\rangle\otimes|\vartheta_\perp\rangle\Big)$$

So, in fact, in the Bell state $|B_{00}\rangle$ the two qubit (or photons) have parallel polarization states but at the same time undetermined or completely random.

In particular if measured in a basis $\{|\vartheta\rangle, |\vartheta_\perp\rangle\}$ in Alice lab the outcome for Alice is prob$(\vartheta) = \frac{1}{2}$ & prob$(\vartheta_\perp) = \frac{1}{2}$. Idem in Bob's lab. We will come back to this issue when we see the concept of density matrix.

II. Teleportation.

Setting



A & B share an entanglement "link" $|B_{00}\rangle$

(qubits 2 & 3 are entangled)

A has a state $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$

(to be teleported to B)

A & B also have a classical communication channel available.

Goal of teleportation protocol:

"Teleport" state $|\varphi\rangle_1$ into Bob's qubit 3

to obtain $|\varphi\rangle_3$.

- Initial state: $|\varphi\rangle_1 \otimes |B_{00}\rangle_{23}$
- Final state: $|\,?\,\rangle_{12} \otimes |\varphi\rangle_3$

we will see that this is $|B_{00}\rangle_{12}$.

Protocol steps (teleportation):

1) A performs on her two qubits (locally in her lab) a Bell basis measurement. The possible outcomes are $|B_{00}\rangle_{12}$, $|B_{01}\rangle_{12}$, $|B_{10}\rangle_{12}$, $|B\rangle_{11_{12}}$

2) A sends to B via the classical channel two classical bits to indicate the outcome

00 , 01 , 10 , 11.

(remark: before receiving this classical message B does not even know a measurement has happened).

3) • If B receives 00 he "applies" $\mathbb{1}_B$ to his qubit (i.e does nothing in fact) and knows he has state $|\varphi\rangle_3 = \alpha|0\rangle_3 + \beta|1\rangle_3$.

• If B receives 01 he knows he has the state $\beta|0\rangle_3 + \alpha|1\rangle_3$ and thus applies X_B to recover $|\varphi\rangle_3$

- If B receives 10 he knows he has the state $\alpha|0\rangle_3 - \beta|1\rangle_3$ and thus applies Z_B to recover $|\varphi\rangle_3$.
- If B receives 11 he knows he has the state $\beta|0\rangle_3 - \alpha|1\rangle_3$ and thus applies $X_B Z_B$ to recover $|\varphi\rangle_3$.

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Analysis of protocol:

- initial state $|\varphi\rangle_1 \otimes |B_{00}\rangle_{23}$
- Measurement of Alice: Imagine she gets out of the meas state $|B_{01}\rangle_{12}$

(other cases are similar and left to you to check).

So the state after the measurement is proportional to:

$$\underbrace{|B_{01}\rangle_{12}\langle B_{01}|_{12}}_{\text{Meas projector in A-lab}} \otimes \mathbb{1}_3 \; |\varphi\rangle_1 \otimes |B_{00}\rangle_{23}.$$

$\uparrow$ indicates B does nothing.

We immediately see the state in A-Lab is $|B_{01}\rangle_{12}$ (first ket). Thus we just have to compute

$$\langle B_{01}|_{12} \otimes \mathbb{1}_3 \; |\varphi\rangle_1 \otimes |B_{00}\rangle_{23} = |?\rangle_3$$

The computation is done by expanding bra-kets :

$$\frac{1}{\sqrt{2}}\left(\langle 0_1\, 1_2| + \langle 1_1\, 0_2|\right)\left(\alpha|0\rangle_1 + \beta|1\rangle_1\right)$$

$$\otimes \frac{1}{\sqrt{2}}\left(|0_2\, 0_3\rangle + |1_2\, 1_3\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\alpha\langle 1_2| + \beta\langle 0_2|\right) \otimes \frac{1}{\sqrt{2}}\left(|0_2\, 0_3\rangle + |1_2\, 1_3\rangle\right)$$

$$= \frac{1}{2}\left(\alpha|1_3\rangle + \beta|0_3\rangle\right).$$

In principle we should normalize this state which consists here in dividing out the $1/2$.

In summary after the A-measurement the global state is

$$|B_{01}\rangle_{12} \otimes \left(\alpha|1\rangle_3 + \beta|0\rangle_3\right).$$

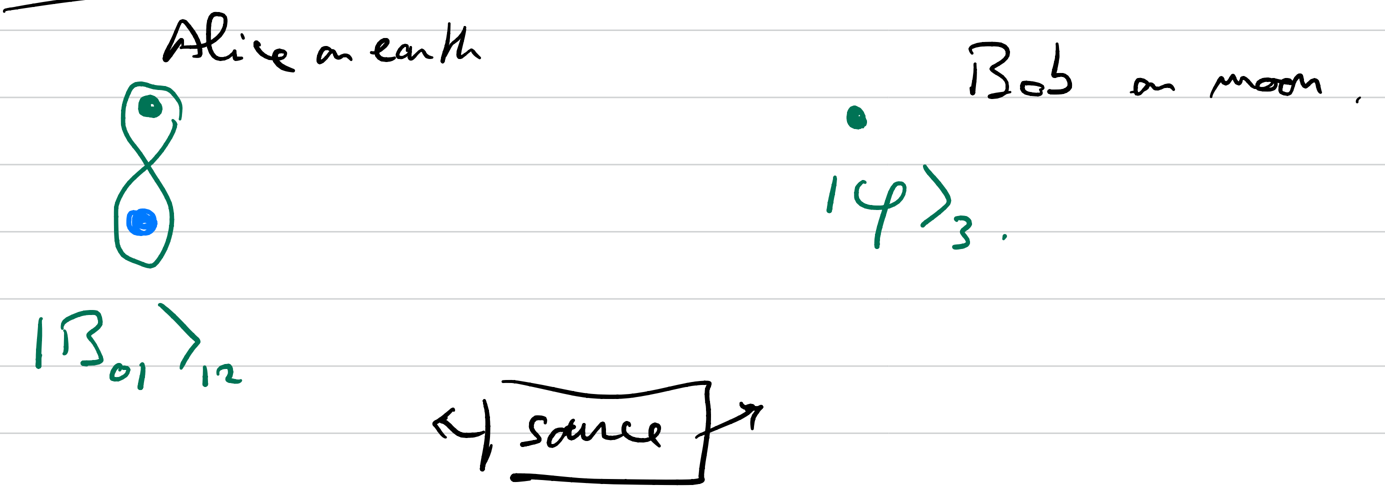
- Alice sends the message 01 to Bob.
- Bob receives 01 and applies X_3 to his qubit. Formally:

$$\underbrace{\mathbb{1}_{12}}_{\text{A-does nothing}} \otimes X_3 \; |B_{01}\rangle_{12} \otimes \left(\alpha |1\rangle_3 + \beta |0\rangle_3 \right)$$

$$= |B_{01}\rangle_{12} \otimes \underbrace{\left(\alpha |0\rangle_3 + \beta |1\rangle_3 \right)}_{|\varphi\rangle_3}.$$

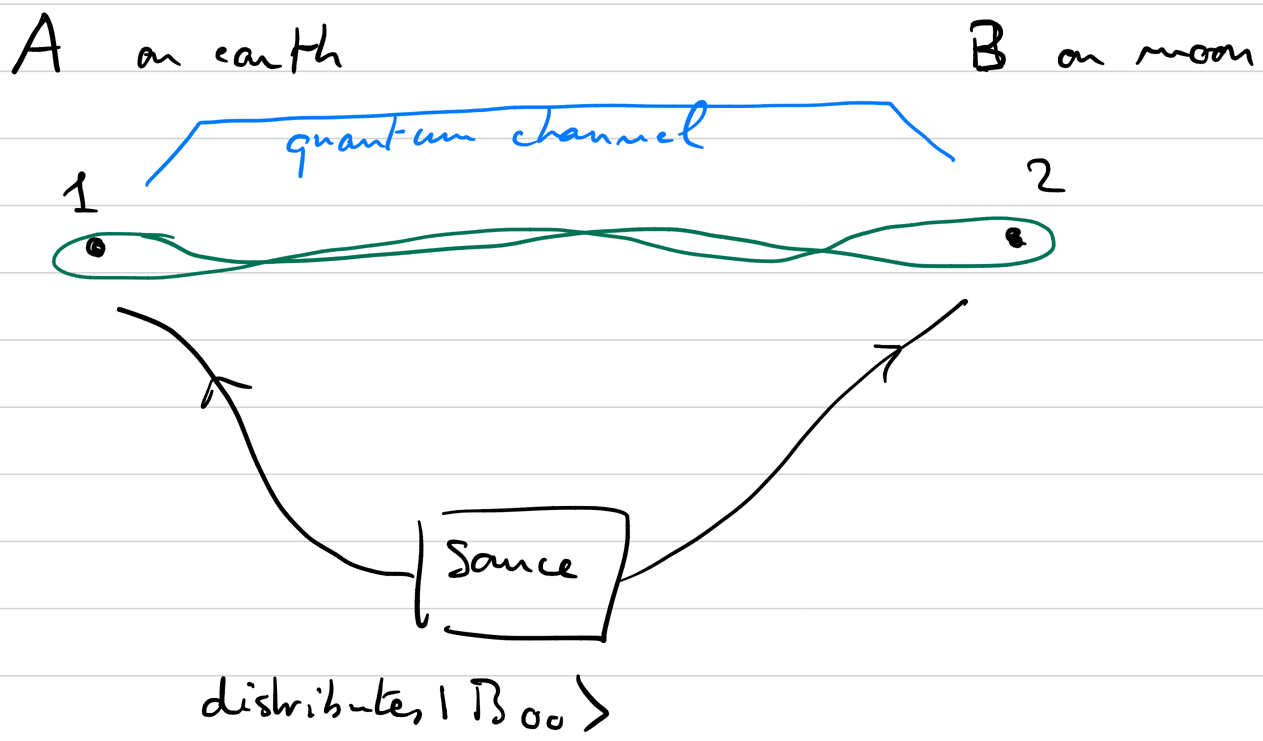
(Recall: $X_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. the "NOT" gate operation).

Final state:



III. Dense coding protocol.

Setting :



- A & B share an entanglement link.
- A & B can communicate via a quantum channel over which qubits are sent (e.g. optic fiber, free space comm ...).

Goal: A wants to convey to B two classical bits by using the quantum channel only once.
"Dense coding"

Protocol steps (dense coding).

- To send 00 A first perform I_A on her qubit (i.e nothing) and sends her qubit over quantum channel.

  Thus Bob receives $|B_{00}\rangle$

- To send 01 A first performs X_A on her qubit and sends her qubit over quantum channel,

  Thus Bob receives $|B_{01}\rangle$

- To send 10 A first performs Z_A on her qubit and sends her qubit over Quantum channel.

  Thus Bob receives $|B_{10}\rangle$

- To send 11 A performs $Z_A X_A$ on her qubit and sends it to Bob. Thus he receives $|B_{11}\rangle$.

- After receiving the qubit of Alice, Bob has a full Bell state:

$|B_{00}\rangle$ or $|B_{01}\rangle$ or $|B_{10}\rangle$ or $|B_{11}\rangle$

He does a measurement in the Bell basis.
Thus by the measurement principle he observes which Bell state he has (important!) i.e the received state is projected to itself here with probability one.

He then decodes to get the classical message as

$$\begin{cases} \text{Meas output } |B_{00}\rangle \to 00 \\ \text{Meas } \text{"} \ |B_{01}\rangle \to 01 \\ \text{Meas } \text{"} \ |B_{10}\rangle \to 10 \\ \text{Meas } \text{"} \ |B_{11}\rangle \to 11. \end{cases}$$

#.

Analysis of protocol:

Suppose Alice wants to send (11). Other cases are similar (exercise).

She first does the unitary op locally in her lab:

$$\underbrace{Z_1 X_1}_{\text{Alice}} \otimes \underbrace{\mathbb{1}_2}_{\substack{\text{Bob does}\\ \text{nothing}}} \quad |B_{00}\rangle_{12}$$

$$= \frac{1}{\sqrt{2}}\left(Z_1 X_1 |0\rangle_1 \otimes |0\rangle_2 + Z_1 X_1 |1\rangle_1 \otimes |1\rangle_2 \right)$$

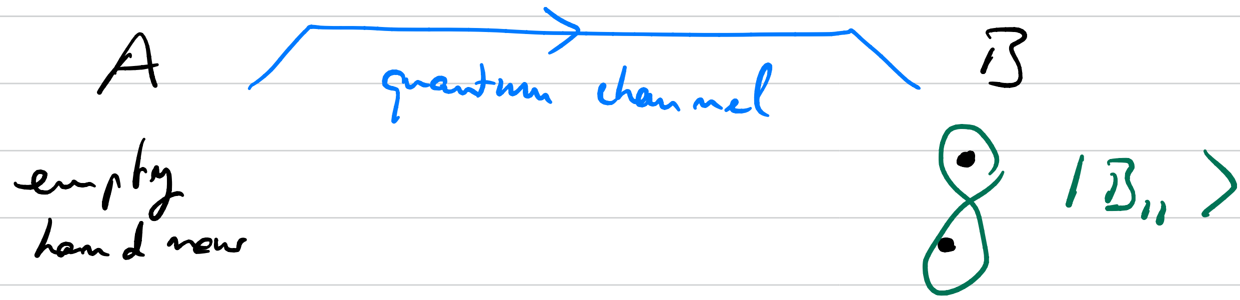
$$= \frac{1}{\sqrt{2}}\left(-|1\rangle_1 \otimes |0\rangle_2 + |0\rangle_1 \otimes |1\rangle_2 \right)$$

$$= -|B_{11}\rangle_{12}$$

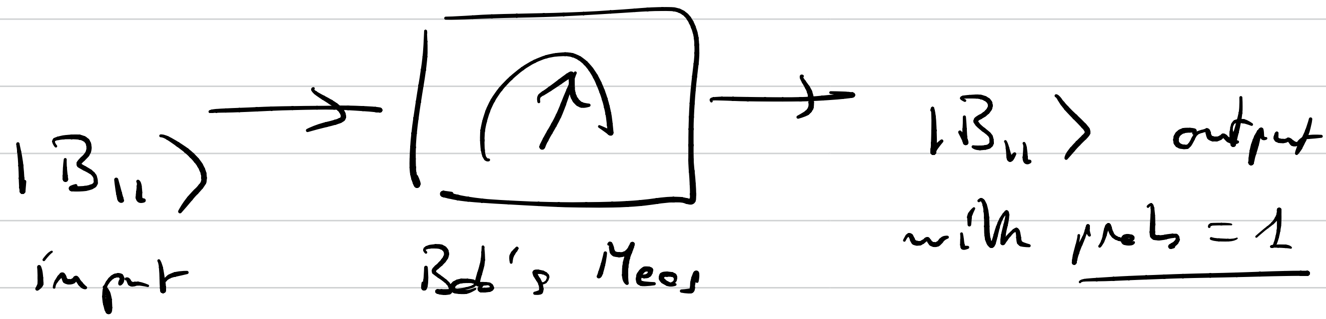
(Global phase (−1) is not important here)

Alice then sends her qubit to Bob over

the quantum channel. Thus Bob get the state $|B_{11}\rangle_{Bob}$.



By the measurement principle for a Bell basis Measurement:



$$\text{prob} = |\underbrace{\langle B_{11}|}_{\text{output}} \underbrace{B_{11}\rangle}_{\text{input}}|^2 = 1$$

<u>**Remark:**</u> $|\langle B_{00}|B_{11}\rangle|^2 = |\langle B_{01}|B_{11}\rangle|^2$

$$= |\langle B_{10}|B_{11}\rangle|^2 = 0$$

Finally Bob knows he got $|B_{11}\rangle$, so he knows the intended classical message is

11.

∎

IV. Final Remarks.

We discuss here two aspects of Bell states.

- How to generate them (in theory)
- How to perform a Bell basis measurement (again in theory).

IV.1 Generation of Bell states.

These can be generated from simple unitary transformations acting on the computational basis states.

The two unitaries involved are

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \& \quad CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

written in the basis $\begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$, $\begin{pmatrix}0\\1\end{pmatrix} = |1\rangle$

and $\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = |00\rangle$, $\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = |01\rangle$, $\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = |10\rangle$,

$\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = |11\rangle$.

<u>Hadamard</u> :

$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

<u>Control NOT</u>

$$\mathrm{CNOT}\,|00\rangle = |00\rangle$$
$$\mathrm{CNOT}\,|01\rangle = |01\rangle$$
$$\mathrm{CNOT}\,|10\rangle = |11\rangle$$
$$\mathrm{CNOT}\,|11\rangle = |10\rangle$$

here $\mathrm{CNOT}\,|b_1\, b_2\rangle = |b_1 \;\; \underbrace{b_2 \oplus b_1}\rangle$

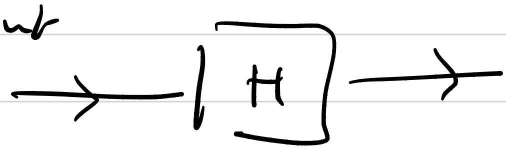
control bit (b_1)   target bit (b_2)

target bit is flipped if control is $b_1 = 1$.

<u>Remark</u>: of course these matrices extend to act on any state by linearity.

Elementary circuit representation

input → output

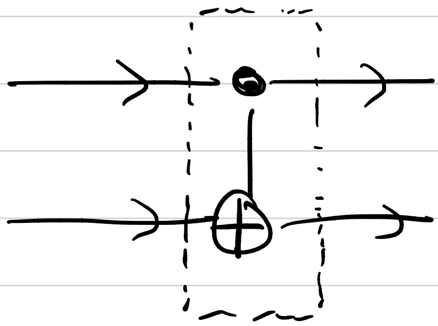


$\alpha|0\rangle + \beta|1\rangle$ $\qquad$ $\alpha \frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$$= \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$$

$|b_1\rangle$ → $|b_1\rangle$

$|b_2\rangle$ → $|b_1 \oplus b_2\rangle$

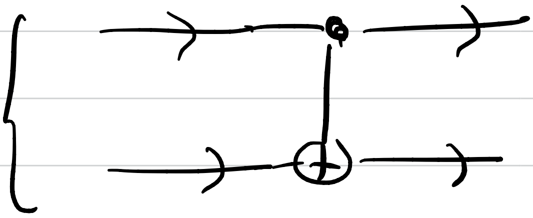


CNOT

and more generally:

input $\qquad$ output

$\sum_{i,j=0}^{1} \alpha_{ij} |i, j\rangle$



$\sum_{i,j=0}^{1} \alpha_{ij} |i, j \oplus i\rangle$

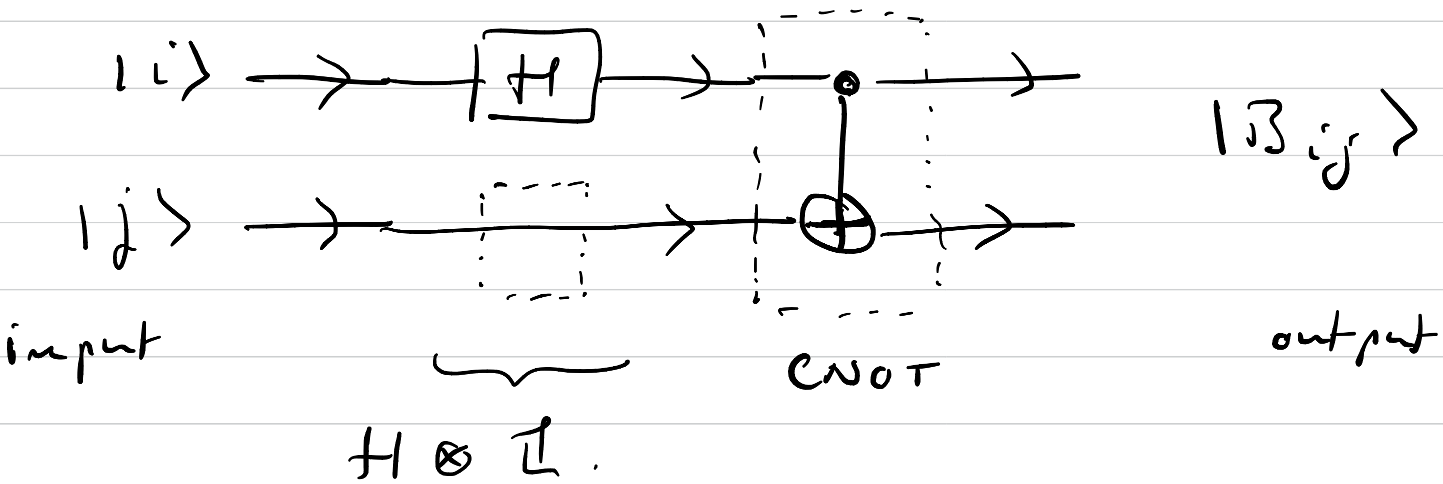
Generation of $|B_{ij}\rangle$ from $|i,j\rangle = |i\rangle \otimes |j\rangle$

One may check the identity:

$$|B_{ij}\rangle = (\text{CNOT})\,(H \otimes \mathbb{1})\,|i\rangle \otimes |j\rangle$$

(exercise)

The circuit representation is then:



[It is the combination of H & CNOT which entangles the two qubits]

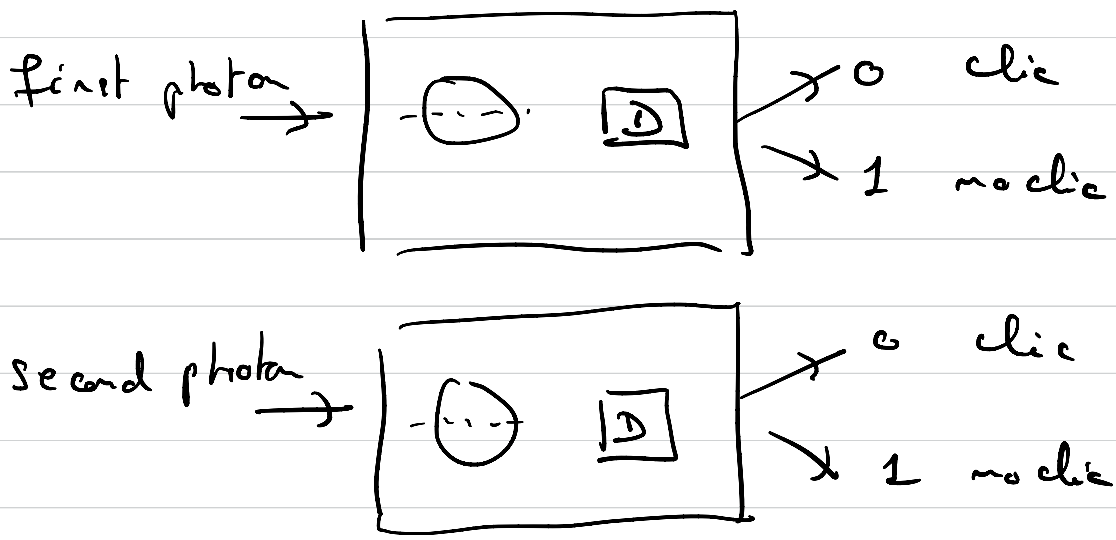
IV. 2 Bell basis measurements.

Suppose we want to measure in a Bell basis $|B_{00}\rangle, |B_{01}\rangle, |B_{10}\rangle, |B_{11}\rangle$ but we only have a measurement apparatus that measures in the computational basis

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$

↑

this could be two "analyzer - detector boxes"



The idea is to implement a basis change before the measurement with the computation basis.

Recall :

$$|B_{ij}\rangle = (\mathrm{CNOT})(H \otimes \mathbb{1})\, |i\rangle \otimes |j\rangle .$$

Now when you measure you project an input state $|\psi\rangle$ as :

$$\propto \underbrace{|B_{ij}\rangle\langle B_{ij}|}_{\substack{\text{projector on} \\ |B_{ij}\rangle}} \psi\rangle \qquad \text{(up to normalization)}$$

"and you register (ij)"

Note that this is equal to :

$$(\mathrm{CNOT})\, H\otimes \mathbb{1}\, \underbrace{|ij\rangle\langle ij|}_{\text{projector on comp basis}} \underbrace{(H\otimes\mathbb{1})\,\mathrm{CNOT}}_{\substack{\text{unitary acting on} \\ \text{input state.}}} |\psi\rangle .$$

Thus it is enough to implement the operation:

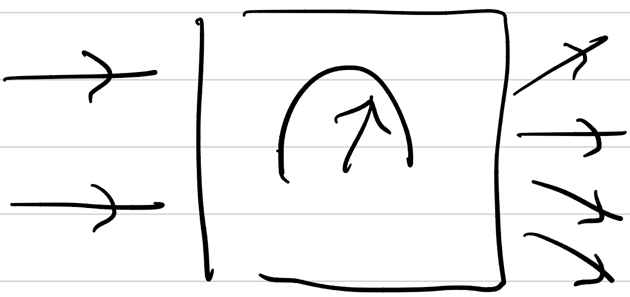
$$\left(|ij\rangle\langle ij|\right)(H\otimes\mathbb{1})\,\mathrm{CNOT}\,|\psi\rangle .$$

and register (i, j) when the state

$$(H \otimes \mathbb{1})\ \text{CNOT}\ |\psi\rangle$$

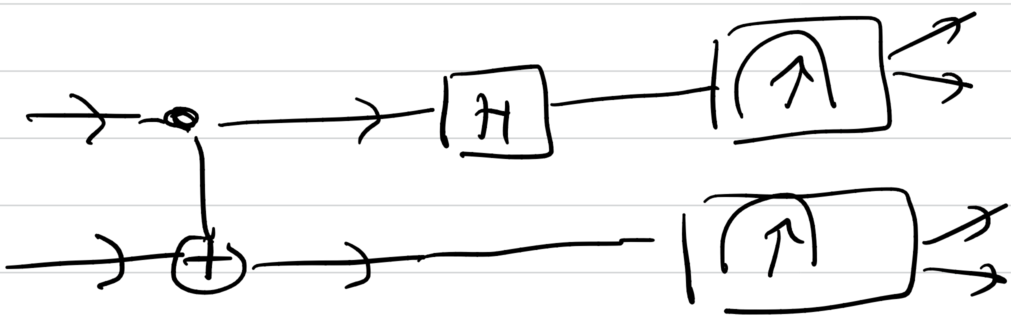
is projected on $|ij\rangle$.

In summary:



Bell basis Measurement

is equivalent to



Computational basis Measurements